

Direct constructions for general families of cyclic mutually nearly orthogonal Latin squares

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Abstract

Two Latin squares $L = [l(i, j)]$ and $M = [m(i, j)]$, of even order n with entries $\{0, 1, 2, \dots, n-1\}$, are said to be nearly orthogonal if the superimposition of L on M yields an $n \times n$ array $A = [(l(i, j), m(i, j))]$ in which each ordered pair (x, y) , $0 \leq x, y \leq n-1$ and $x \neq y$, occurs at least once and the ordered pair $(x, x+n/2)$ occurs exactly twice. In this paper, we present direct constructions for the existence of general families of three cyclic mutually orthogonal Latin squares of orders $48k+14$, $48k+22$, $48k+38$ and $48k+46$. The techniques employed are based on the principle of Methods of Differences and so we also establish infinite classes of “quasi-difference” sets for these orders.

Keywords: Latin squares, orthogonal Latin squares, nearly orthogonal Latin squares, quasi-difference sets

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1 Introduction

A *Latin square*, $L = [l(i, j)]$, of order n is an $n \times n$ array in which each row and each column contains each of the symbols $0, 1, \dots, n-1$ precisely once. Given two Latin squares $L = [l(i, j)]$ and $M = [m(i, j)]$, of order n , we define the superimposition of L on M to be the $n \times n$ array $A = [(l(i, j), m(i, j))]$, so the cell (i, j) of A contains the ordered pair $(l(i, j), m(i, j))$. The Latin squares L and M are said to be *orthogonal* if each of the ordered pair (x, y) , $0 \leq x, y \leq n-1$, occurs in a cell of A . A set of s *mutually orthogonal Latin squares* (MOLS(n)) is a set of s Latin squares which are pairwise orthogonal.

Orthogonal Latin squares have wide ranging applications and have consequently been studied with great interest. However, there are still many open questions relating to their existences. For instance, it is known that there does not exist a pair of MOLS(6), however it is not known if there exists a set of three MOLS(10), see [1]. In 2012, Todorov established that there exists a set of four MOLS(14), but it is not known if there exists a set of five MOLS(14), see [5]. The order 22 is the largest order for which it is not known if there exists a set of four MOLS(22).

In 2002, Raghavarao, Shrikhande and Shrikhande suggested [4] that given the importance of their applications in experimental design, the definition of MOLSs could be varied slightly to deal with orders for which MOLSs are not known to exist. They suggested that the orthogonality condition could be adapted in such a way that identical pairs did not occur, n specified pairs occurred twice and all other pairs occurred precisely once.

Two Latin squares $L = [l(i, j)]$ and $M = [m(i, j)]$, of even order n , are said to be *nearly orthogonal* [4] if the superimposition of L on M yields an $n \times n$ array $A = [(l(i, j), m(i, j))]$ in which each ordered pair (x, y) , $0 \leq x, y \leq n-1$ and $x \neq y$, occurs at least once and the ordered pair $(x, x + n/2)$ occurs exactly twice. As a consequence of the definition, we note that none of the n ordered pairs (x, x) , $0 \leq x \leq n-1$, occurs in A . A set of s *mutually nearly orthogonal Latin squares* (MNOLS(n)) is a set of s Latin squares which are pairwise nearly orthogonal.

It is known that there exist a set of three MNOLS(6), a set of four MNOLS(10) and a set of four MNOLS(14), but no set of four MNOLS(6), see [3, 4], raising interesting questions about the existence of sets of MNOLS(n).

Raghavarao, Shrikhande and Shrikhande, established the following upper bound on the size of a set of MNOLSs of order n .

Theorem 1.1. [4] *Let L_1, L_2, \dots, L_t be t Latin squares of order $n = 2m$ on symbols $\{0, 1, 2, \dots, n-1\}$ such that each pair of Latin squares is nearly orthogonal. Then*

$$t \leq \begin{cases} \frac{n}{2} + 1, & \text{if } n \equiv 2 \pmod{4}, \text{ or,} \\ \frac{n}{2}, & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

In the paper [4], Raghavarao, Shrikhande and Shrikhande used the principle of the Method of Differences to established a construction for MNOLSs:

Theorem 1.2. [4] *Let there exist t column vectors of length $2m$, denoted $C_s = [c_s(i, 0)]$, for $0 \leq i \leq 2m-1$ and $s = 1, 2, \dots, t$, where each column vector is a*

permutation of the elements of the cyclic group \mathbb{Z}_{2m} . Furthermore, suppose for every $s \neq s'$, $1 \leq s, s' \leq t$, among the $2m$ differences $c_s(1, 0) - c_{s'}(1, 0), c_s(2, 0) - c_{s'}(2, 0), \dots, c_s(2m-1, 0) - c_{s'}(2m-1, 0)$ modulo $2m$, m occurs twice and all other non-zero elements of \mathbb{Z}_{2m} occur once. Then $\mathcal{L}_s = [l_s(i, j)]$, where $l_s(i, j) \equiv (c_s(i, 0) + j)(\text{mod } 2m)$ for $0 \leq j \leq 2m-1$ and $s = 1, 2, \dots, t$, forms a set of t MNOLS($2m$).

The MNOLSs, \mathcal{L}_s , constructed as in Theorem 1.2 will be termed *cyclic* MNOLSs. In [3], it was proven that there exist two cyclic MNOLSs of order $2m$ for all $m \geq 2$. In the same paper, it was also proven that there exist three MNOLS($2m$) for all $2m \geq 358$. But the existence of three cyclic MNOLSs of order $2m$ is still open.

In this paper, we prove the existence of general families of column vectors which establish the existence of three cyclic MNOLSs of orders $48k+14$, $48k+22$, $48k+38$ and $48k+46$ for all $k \in \mathbb{Z}^+ \cup \{0\}$. Since the constructions are based on the principle of Methods of Differences the paper also establishes infinite classes of “quasi-difference” sets for these orders, which may have applications in the theory of orthomorphisms, see [6, 2].

The Latin squares generated here will be of even order and cyclic. In addition, they will all have the following property. We will say that the column vector \mathcal{C} has the *reflection property*, if $c(i, 0) + c(n-1-i, 0) \equiv n-1 \pmod{n}$ for all $i = 0, \dots, (n-2)/2$. Further we will say that MNOLSs developed from such column vector, also have the reflection property.

Example 1.3. Let $V_1 = \{(i, 0, i) \mid 0 \leq i \leq (n-2)/2\}$ and $\overline{V}_1 = \{(n-1-i, 0, n-1-i) \mid 0 \leq i \leq (n-2)/2\}$. Then $\mathcal{C}_1 = V_1 \cup \overline{V}_1$ has the reflection property. Let $\mathcal{L}_1 = [l_1(i, j)]$, where $l_1(i, j) \equiv (c_1(i, 0) + j)(\text{mod } n)$ for $0 \leq j \leq n-1$. Then \mathcal{L}_1 is also said to have the reflection property.

In subsequent sections, the symbol \otimes has been used to represent “a contradiction”.

2 Three cyclic MNOLSs of Order $48k+14$, $k \geq 0$

In this section we construct two cyclic Latin squares \mathcal{L}_2 and \mathcal{L}_3 both of order $48k+14$ and show that \mathcal{L}_1 (constructed by Example 1.3 with $n = 48k+14$), \mathcal{L}_2 and \mathcal{L}_3 are cyclic MNOLSs. The following lemma is crucial in this section.

Lemma 2.1. Let k be an integer. Working modulo $48k+14$, **1.** $\gcd(6k+2, 24k+7) = 1$; **2.** $\gcd(12k+5, 48k+14) = 1$; **3.** $\gcd(6k+1, 24k+7) = 1$; **4.** $\gcd(12k+3, 48k+14) = 1$.

Proof. The following equations verify the statements given in the lemma: **1.** $4(6k+2) - (24k+7) = 1$; **2.** $(8k+3)(12k+5) - (2k+1)(48k+14) = 1$; **3.** $(4k+1)(24k+7) - (16k+6)(6k+1) = 1$; **4.** $(24k+5)(12k+3) - (6k+1)(48k+14) = 1$. \square

Working modulo $48k+14$ we define the $(24k+7) \times 1$ matrices (column vectors)

$V_\alpha = [v_\alpha(i, 0)]$, $\alpha = 2, 3$, by

$$\begin{aligned} V_2 = & \{(2i, 0, 6k + 1 + i(12k + 4)) \mid 0 \leq i \leq 12k + 3\} \cup \\ & \{(2i + 1, 0, 12k + 3 + i(12k + 4)) \mid 0 \leq i \leq 12k + 2\}, \end{aligned} \quad (1)$$

$$\begin{aligned} V_3 = & \{(2i, 0, 6k + 2 + i(12k + 5)) \mid 0 \leq i \leq 12k + 3\} \cup \\ & \{(2i + 1, 0, 24k + 8 + i(12k + 5)) \mid 0 \leq i \leq 12k + 2\}. \end{aligned} \quad (2)$$

For $\alpha = 2, 3$, let $\mathcal{C}_\alpha = V_\alpha \cup \overline{V_\alpha}$, where

$$\overline{V_\alpha} = \{(48k + 13 - i, 0, 48k + 13 - v_\alpha(i, 0)) \mid 0 \leq i \leq 24k + 6\}.$$

Note that \mathcal{C}_α has the reflection property. Now define $\mathcal{L}_\alpha = [l_\alpha(i, j)]$, where $l_\alpha(i, j) \equiv \mathcal{C}_\alpha(i, 0) + j \pmod{48k + 14}$ for $0 \leq i, j \leq 48k + 13$.

Lemma 2.2. *The array \mathcal{L}_2 is a Latin square of order $48k + 14$, $k \geq 0$.*

Proof. The entries in V_2 are all distinct as verified by Equation 3 for the case rows $2i$ and $2j$, where $0 \leq i, j \leq 12k + 3$, Equation 4 for the case rows $2i + 1$ and $2j + 1$, where $0 \leq i, j \leq 12k + 2$, and Equation 5 for the case rows $2i$ and $2j + 1$, where $0 \leq i \leq 12k + 3$ and $0 \leq j \leq 12k + 2$.

$$\begin{aligned} 6k + 1 + i(12k + 4) & \equiv 6k + 1 + j(12k + 4) \pmod{48k + 14}, \\ \Rightarrow (j - i)(6k + 2) & \equiv 0 \pmod{24k + 7}, \not\equiv; \end{aligned} \quad (3)$$

$$\begin{aligned} 12k + 3 + i(12k + 4) & \equiv 12k + 3 + j(12k + 4) \pmod{48k + 14}, \\ \Rightarrow (j - i)(6k + 2) & \equiv 0 \pmod{24k + 7}, \not\equiv; \end{aligned} \quad (4)$$

$$\begin{aligned} 6k + 1 + i(12k + 4) & \equiv 12k + 3 + j(12k + 4) \pmod{48k + 14}, \\ \Rightarrow (j - i)(6k + 2) & \equiv -3k - 1 \pmod{24k + 7}, \\ \Rightarrow j - i & \equiv 4(-3k - 1) \pmod{24k + 7}, \end{aligned} \quad (5)$$

implying $j - i = 12k + 3$, or $j = 12k + 3 + i > 12k + 2$, which leads to a contradiction.

For any two rows containing entries x and y in V_2 , parity conditions and the following equations can be used to verify $x + y + 1 \not\equiv 0 \pmod{48k + 14}$, specifically Equation 6 for rows $2i$ and $2j$, Equation 7 for rows $2i + 1$ and $2j + 1$ and Equation 8 for rows $2i$ and $2j + 1$.

$$\begin{aligned} 2(6k + 1) + (i + j)(12k + 4) + 1 & \equiv 0 \pmod{48k + 14}, \\ \Rightarrow 12k + 3 + (i + j)(12k + 4) & \equiv 0 \pmod{48k + 14}, \not\equiv; \end{aligned} \quad (6)$$

$$\begin{aligned} 2(12k + 3) + (i + j)(12k + 4) + 1 & \equiv 0 \pmod{48k + 14}, \\ \Rightarrow 24k + 7 + (i + j)(12k + 4) & \equiv 0 \pmod{48k + 14}, \not\equiv; \end{aligned} \quad (7)$$

$$\begin{aligned} 6k + 1 + 12k + 3 + (i + j)(12k + 4) + 1 & \equiv 0 \pmod{48k + 14}, \\ \Rightarrow 18k + 5 + (i + j)(12k + 4) & \equiv 0 \pmod{48k + 14}, \not\equiv. \end{aligned} \quad (8)$$

Thus the entries of \mathcal{C}_2 are all distinct and so \mathcal{L}_2 is a Latin square of order $48k + 14$. \square

Lemma 2.3. *The array \mathcal{L}_3 is a Latin square of order $48k + 14$, $k \geq 0$.*

Proof. The entries in V_3 are all distinct as verified by Equation 9 for the case rows $2i$ and $2j$, where $0 \leq i, j \leq 12k + 3$, Equation 10 for the case rows $2i + 1$ and $2j + 1$, where $0 \leq i, j \leq 12k + 2$, and Equation 11 for the case rows $2i$ and $2j + 1$, where $0 \leq i \leq 12k + 3$ and $0 \leq j \leq 12k + 2$.

$$\begin{aligned} 6k + 2 + i(12k + 5) &\equiv 6k + 2 + j(12k + 5) \pmod{48k + 14}, \\ \Rightarrow (j - i)(12k + 5) &\equiv 0 \pmod{48k + 14}, \otimes; \end{aligned} \quad (9)$$

$$\begin{aligned} 24k + 8 + i(12k + 5) &\equiv 24k + 8 + j(12k + 5) \pmod{48k + 14}, \\ \Rightarrow (j - i)(12k + 5) &\equiv 0 \pmod{48k + 14}, \otimes; \end{aligned} \quad (10)$$

$$\begin{aligned} 6k + 2 + i(12k + 5) &\equiv 24k + 8 + j(12k + 5) \pmod{48k + 14}, \\ j - i &\equiv (-18k - 6)(8k + 3) \equiv 36k + 10 \pmod{48k + 14}, \end{aligned} \quad (11)$$

implying $j = 36k + 10 + i > 12k + 2$ or $j = -12k - 4 + i < 0$, which leads to a contradiction.

For any two rows containing entries x and y in V_3 , $x + y + 1 \not\equiv 0 \pmod{48k + 14}$ as verified by Equation 12 for rows $2i$ and $2j$, Equation 13 for rows $2i + 1$ and $2j + 1$ and Equation 14 for rows $2i$ and $2j + 1$.

$$\begin{aligned} 2(6k + 2) + (i + j)(12k + 5) + 1 &\equiv 0 \pmod{48k + 14}, \\ \Rightarrow (i + j + 1)(12k + 5) &\equiv 0 \pmod{48k + 14}, \\ \Rightarrow i + j + 1 &\equiv 0 \pmod{48k + 14}, \otimes; \end{aligned} \quad (12)$$

$$\begin{aligned} 2(24k + 8) + (i + j)(12k + 5) + 1 &\equiv 0 \pmod{48k + 14}, \\ \Rightarrow (i + j)(12k + 5) &\equiv -48k - 17 \pmod{48k + 14}, \\ \Rightarrow i + j &\equiv (-48k - 17)(8k + 3) \equiv 24k + 5 \pmod{48k + 14} \otimes; \end{aligned} \quad (13)$$

$$\begin{aligned} 6k + 2 + 24k + 8 + (i + j)(12k + 5) + 1 &\equiv 0 \pmod{48k + 14}, \\ \Rightarrow (i + j)(12k + 5) &\equiv -30k - 11 \pmod{48k + 14}, \\ \Rightarrow i + j &\equiv (-30k - 11)(8k + 3) \equiv 36k + 9 \pmod{48k + 14}, \otimes. \end{aligned} \quad (14)$$

Thus the entries of \mathcal{C}_3 are all distinct and so \mathcal{L}_3 is a Latin square of order $48k + 14$. \square

Theorem 2.4. *The Latin squares \mathcal{L}_1 , \mathcal{L}_2 and \mathcal{L}_3 are cyclic MNOLSs of order $48k + 14$, $k \geq 0$.*

Proof. Respectively, the differences between entries in rows $2i$ and $2i + 1$ of V_2 and V_1 , are

$$\begin{aligned} 6k + 1 + i(12k + 4) - 2i &\equiv (2i + 1)(6k + 1) \pmod{48k + 14}, \\ 12k + 3 + i(12k + 4) - 2i - 1 &\equiv (2i + 2)(6k + 1) \pmod{48k + 14}. \end{aligned}$$

These differences are all non-zero since in the first instance $(2i + 1)(6k + 1)$ is odd and $48k + 14$ is even and in the second instance if the difference $(2i + 2)(6k + 1) \equiv 0 \pmod{48k + 14}$, then by Lemma 2.1, $i + 1 \equiv 0 \pmod{24k + 7}$, which implies $i = 24k + 6$, a contradiction.

The differences are all distinct as verified by Equation 15 for rows $2i$ and $2j$ and for rows $2i + 1$ and $2j + 1$, and using a parity argument in Equation 16 for rows $2i$ and $2j + 1$.

$$\begin{aligned} 2(j - i)(6k + 1) &\equiv 0 \pmod{48k + 14}, \\ \Rightarrow (j - i)(6k + 1) &\equiv 0 \pmod{24k + 7}, \not\equiv; \end{aligned} \quad (15)$$

$$2(j - i)(6k + 1) \equiv -6k - 1 \pmod{48k + 14}, \not\equiv. \quad (16)$$

In addition, any two distinct differences x and y , produced by corresponding rows of V_2 and V_1 , satisfy $x + y \not\equiv 0 \pmod{48k + 14}$, as verified by Equation 17 for rows $2i$ and $2j$, Equation 18 for rows $2i + 1$ and $2j + 1$ and parity arguments together with Equation 19 for rows $2i$ and $2j + 1$. In all such cases $x + y$ is congruent to

$$\begin{aligned} (i + j + 1)(6k + 1) &\equiv 0 \pmod{24k + 7}, \\ \Rightarrow i + j + 1 &\equiv 0 \pmod{24k + 7}, \not\equiv; \end{aligned} \quad (17)$$

$$\begin{aligned} (i + j + 2)(6k + 1) &\equiv 0 \pmod{24k + 7}, \\ \Rightarrow i + j + 2 &\equiv 0 \pmod{24k + 7}, \not\equiv; \end{aligned} \quad (18)$$

$$6k + 1 + 2(i + j + 1)(6k + 1) \equiv 0 \pmod{48k + 14}, \not\equiv. \quad (19)$$

Respectively, the differences between entries in rows $2i$ and $2i + 1$ of V_3 and V_1 , are

$$\begin{aligned} 6k + 2 + i(12k + 5) - 2i &\equiv 6k + 2 + i(12k + 3) \pmod{48k + 14}, \\ 24k + 8 + i(12k + 5) - 2i - 1 &\equiv 24k + 7 + i(12k + 3) \pmod{48k + 14}. \end{aligned}$$

Equations 20 and 21 verify that these differences are all non-zero.

$$i \equiv (-6k - 2)(24k + 5) \equiv 12k + 4 \pmod{48k + 14}, \not\equiv \quad (20)$$

$$i \equiv (-24k - 7)(24k + 5) \equiv 24k + 7 \pmod{48k + 14}, \not\equiv. \quad (21)$$

If two differences produced by rows $2i$ and $2j$ or by rows $2i + 1$ and $2j + 1$ are equal, then $(j - i)(12k + 3) \equiv 0 \pmod{48k + 14}$. Now by Lemma 2.1, $i = j$. Equation 22 verifies that two differences produced by rows $2i$ and $2j + 1$ are never equal.

$$\begin{aligned} (j - i)(12k + 3) &\equiv -18k - 5 \pmod{48k + 14}, \\ \Rightarrow j - i \equiv (-18k - 5)(24k + 5) &\equiv 12k + 3 \pmod{48k + 14}, \end{aligned} \quad (22)$$

implying $j = i + 12k + 3 > 12k + 2$, which leads to a contradiction.

In addition, any two distinct differences x and y , produced by corresponding rows of V_3 and V_1 , satisfy $x + y \not\equiv 0 \pmod{48k + 14}$, as verified by Equation 23 for rows $2i$ and $2j$, Equation 24 for rows $2i + 1$ and $2j + 1$ and parity arguments together with Equation 25 for rows $2i$ and $2j + 1$. In all such cases $x + y$ is congruent to

$$\begin{aligned} 12k + 4 + (i + j)(12k + 3) &\equiv 0 \pmod{48k + 14}, \\ \Rightarrow i + j \equiv (-12k - 4)(24k + 5) &\equiv 24k + 8 \pmod{48k + 14}, \not\equiv; \end{aligned} \quad (23)$$

$$(i + j)(12k + 3) \equiv 0 \pmod{48k + 14}, \not\equiv; \quad (24)$$

$$\begin{aligned} 30k + 9 + (i + j)(12k + 3) &\equiv 0 \pmod{48k + 14}, \\ \Rightarrow i + j \equiv (-30k - 9)(24k + 5) &\equiv 36k + 11 \pmod{48k + 14}, \not\equiv. \end{aligned} \quad (25)$$

Respectively, the differences between entries are in rows $2i$ and $2i + 1$ of V_3 and V_2 , are

$$\begin{aligned} 6k + 2 - 6k - 1 + i(12k + 5 - 12k - 4) &\equiv i + 1 \pmod{48k + 14}, \\ 24k + 8 - 12k - 3 + i(12k + 5 - 12k - 4) &\equiv 12k + 5 + i \pmod{48k + 14}. \end{aligned}$$

These are all non-zero and distinct. In addition, any two distinct differences x and y satisfy $x + y \not\equiv 0 \pmod{48k + 14}$.

By Lemmas 2.2, 2.3 and the above arguments, the Latin squares \mathcal{L}_1 , \mathcal{L}_2 and \mathcal{L}_3 are cyclic MNOLSs. \square

3 Three cyclic MNOLSs of Order $48k + 22$, $k \geq 0$

In this section we construct two cyclic Latin squares \mathcal{L}_2 and \mathcal{L}_3 both of order $48k + 22$ and show that \mathcal{L}_1 (constructed by Example 1.3 with $n = 48k + 22$), \mathcal{L}_2 and \mathcal{L}_3 are cyclic MNOLSs. The following lemma is crucial in this section.

Lemma 3.1. *Let k be an integer. Working modulo $48k + 22$, **1.** $\gcd(6k + 3, 24k + 11) = 1$; **2.** $\gcd(12k + 7, 48k + 22) = 1$; **3.** $\gcd(6k + 2, 24k + 11) = 1$; **4.** $\gcd(12k + 5, 48k + 22) = 1$.*

Proof. The following equations verify the statements given in the lemma: **1.** $4(6k + 3) - (24k + 11) = 1$; **2.** $(2k + 1)(48k + 22) - (8k + 3)(12k + 7) = 1$; **3.** $(2k + 1)(24k + 11) - (8k + 5)(6k + 2) = 1$; **4.** $(24k + 9)(12k + 5) - (6k + 2)(48k + 22) = 1$. \square

Working modulo $48k + 22$ we define the $(24k + 11) \times 1$ matrices (column vectors) $V_\alpha = [v_\alpha(i, 0)]$, $\alpha = 2, 3$, by

$$\begin{aligned} V_2 &= \{(2i, 0, 30k + 13 + i(12k + 6)) \mid 0 \leq i \leq 12k + 5\} \cup \\ &\quad \{(2i + 1, 0, 12k + 5 + i(12k + 6)) \mid 0 \leq i \leq 12k + 4\}, \end{aligned} \quad (26)$$

$$\begin{aligned} V_3 &= \{(2i, 0, 30k + 14 + i(12k + 7)) \mid 0 \leq i \leq 12k + 5\} \cup \\ &\quad \{(2i + 1, 0, 24k + 12 + i(12k + 7)) \mid 0 \leq i \leq 12k + 4\}. \end{aligned} \quad (27)$$

For $\alpha = 2, 3$, let $\mathcal{C}_\alpha = V_\alpha \cup \overline{V_\alpha}$, where

$$\overline{V_\alpha} = \{(48k + 21 - i, 0, 48k + 21 - v_\alpha(i, 0)) \mid 0 \leq i \leq 24k + 10\}.$$

Note that \mathcal{C}_α has the reflection property. Now define $\mathcal{L}_\alpha = [l_\alpha(i, j)]$, where $l_\alpha(i, j) \equiv \mathcal{C}_\alpha(i, 0) + j \pmod{48k + 22}$ for $0 \leq i, j \leq 48k + 21$.

Lemma 3.2. *The array \mathcal{L}_2 is a Latin square of order $48k + 22$, for $k \geq 0$.*

Proof. The entries in V_2 are all distinct as verified by Equation 28 for the case rows $2i$ and $2j$, where $0 \leq i, j \leq 12k + 5$, Equation 29 for the case rows $2i + 1$ and $2j + 1$,

where $0 \leq i, j \leq 12k + 4$, and Equation 30 for the case rows $2i$ and $2j + 1$, where $0 \leq i \leq 12k + 5$ and $0 \leq j \leq 12k + 4$.

$$\begin{aligned} 30k + 13 + i(12k + 6) &\equiv 30k + 13 + j(12k + 6) \pmod{48k + 22}, \\ \Rightarrow (j - i)(6k + 3) &\equiv 0 \pmod{24k + 11}, \otimes; \end{aligned} \quad (28)$$

$$\begin{aligned} 12k + 5 + i(12k + 6) &\equiv 12k + 5 + j(12k + 6) \pmod{48k + 22} \\ \Rightarrow (j - i)(6k + 3) &\equiv 0 \pmod{24k + 11}, \otimes; \end{aligned} \quad (29)$$

$$\begin{aligned} 30k + 13 + i(12k + 6) &\equiv 12k + 5 + j(12k + 6) \pmod{48k + 22}, \\ \Rightarrow (j - i)(6k + 3) &\equiv 9k + 4 \pmod{24k + 11}, \\ \Rightarrow j - i &\equiv 4(9k + 4) \pmod{24k + 11}, \otimes. \end{aligned} \quad (30)$$

For any two rows containing entries x and y in V_2 , parity conditions and the following equations can be used to verify that $x + y + 1 \not\equiv 0 \pmod{48k + 22}$, specifically Equation 31 for rows $2i$ and $2j$, Equation 32 for rows $2i + 1$ and $2j + 1$ and Equation 33 for rows $2i$ and $2j + 1$.

$$\begin{aligned} 2(30k + 13) + (i + j)(12k + 6) + 1 &\equiv 0 \pmod{48k + 22}, \\ \Rightarrow 12k + 5 + (i + j)(12k + 6) &\equiv 0 \pmod{48k + 22}, \otimes; \end{aligned} \quad (31)$$

$$\begin{aligned} 2(12k + 5) + (i + j)(12k + 6) + 1 &\equiv 0 \pmod{48k + 22}, \\ \Rightarrow 24k + 11 + (i + j)(12k + 6) &\equiv 0 \pmod{48k + 22}, \otimes; \end{aligned} \quad (32)$$

$$\begin{aligned} 30k + 13 + 12k + 5 + (i + j)(12k + 6) + 1 &\equiv 0 \pmod{48k + 22}, \\ \Rightarrow 42k + 19 + (i + j)(12k + 6) &\equiv 0 \pmod{48k + 22}, \otimes. \end{aligned} \quad (33)$$

Thus the entries of \mathcal{C}_2 are all distinct and so \mathcal{L}_2 is a Latin square of order $48k + 22$. \square

Lemma 3.3. *The array \mathcal{L}_3 is a Latin square of order $48k + 22$, for $k \geq 0$.*

Proof. The entries in V_3 are all distinct as verified by Equation 34 for the case rows $2i$ and $2j$, where $0 \leq i, j \leq 12k + 5$, Equation 35 for the case rows $2i + 1$ and $2j + 1$, where $0 \leq i, j \leq 12k + 4$, and Equation 36 for the case rows $2i$ and $2j + 1$, where $0 \leq i \leq 12k + 5$ and $0 \leq j \leq 12k + 4$.

$$\begin{aligned} 30k + 14 + i(12k + 7) &\equiv 30k + 14 + j(12k + 7) \pmod{48k + 22}, \\ \Rightarrow (j - i)(12k + 7) &\equiv 0 \pmod{48k + 22}, \otimes; \end{aligned} \quad (34)$$

$$\begin{aligned} 24k + 12 + i(12k + 7) &\equiv 24k + 12 + j(12k + 7) \pmod{48k + 22}, \\ \Rightarrow (j - i)(12k + 7) &\equiv 0 \pmod{48k + 22}, \otimes; \end{aligned} \quad (35)$$

$$\begin{aligned} 30k + 14 + i(12k + 7) &\equiv 24k + 12 + j(12k + 7) \pmod{48k + 22}, \\ \Rightarrow (j - i)(12k + 7) &\equiv 6k + 2 \pmod{48k + 22}, \\ \Rightarrow j - i \equiv (-8k - 3)(6k + 2) &\equiv 36k + 16 \pmod{48k + 22}, \end{aligned} \quad (36)$$

implying $j = 36k + 16 + i > 12k + 4$ or $j = -12k - 6 + i < 0$, a contradiction.

For any two rows containing entries x and y in V_3 , $x + y + 1 \not\equiv 0 \pmod{48k + 22}$ as verified by Equation 37 for rows $2i$ and $2j$, Equation 38 for rows $2i + 1$ and $2j + 1$

and Equation 39 for rows $2i$ and $2j + 1$.

$$\begin{aligned} 2(30k + 14) + (i + j)(12k + 7) + 1 &\equiv 0 \pmod{48k + 22}, \\ \Rightarrow (i + j + 1)(12k + 7) &\equiv 0 \pmod{48k + 22}, \not\equiv; \end{aligned} \quad (37)$$

$$\begin{aligned} 2(24k + 12) + (i + j)(12k + 7) + 1 &\equiv 0 \pmod{48k + 22}, \\ \Rightarrow (i + j)(12k + 7) &\equiv -3 \pmod{48k + 22}, \\ \Rightarrow i + j &\equiv (-3)(-8k - 3) \equiv 24k + 9 \pmod{48k + 22}, \not\equiv; \end{aligned} \quad (38)$$

$$\begin{aligned} 30k + 14 + 24k + 12 + (i + j)(12k + 7) + 1 &\equiv 0 \pmod{48k + 22}, \\ \Rightarrow (i + j)(12k + 7) &\equiv -6k - 5 \pmod{48k + 22}, \\ \Rightarrow i + j &\equiv (-6k - 5)(-8k - 3) \equiv 36k + 15 \pmod{48k + 22}, \not\equiv \end{aligned} \quad (39)$$

Thus the entries of \mathcal{C}_3 are all distinct and so \mathcal{L}_3 is a Latin square of order $48k + 22$. \square

Theorem 3.4. *The Latin squares \mathcal{L}_1 , \mathcal{L}_2 and \mathcal{L}_3 are cyclic MNOLSs of order $48k + 22$, $k \geq 0$.*

Proof. Respectively, for rows $2i$ and $2i + 1$ the differences between entries of V_2 and V_1 , are

$$\begin{aligned} 30k + 13 + i(12k + 6) - 2i &\equiv 30k + 13 + i(12k + 4) \pmod{48k + 22}, \\ 12k + 5 + i(12k + 6) - 2i - 1 &\equiv 12k + 4 + i(12k + 4) \pmod{48k + 22}. \end{aligned}$$

These differences are all non-zero because in the first instance since $30k + 13$ is odd but $12k + 4$ and $48k + 22$ are even and in the second if $(i + 1)(12k + 4) \equiv 0 \pmod{48k + 22}$, then $i + 1 \equiv 0 \pmod{24k + 11}$, implying $i = 24k + 10$, a contradiction.

The differences are all distinct as verified by Equation 40 for rows $2i$ and $2j$ and for rows $2i + 1$ and $2j + 1$, and using a parity argument in Equation 41 for rows $2i$ and $2j + 1$.

$$\begin{aligned} (j - i)(12k + 4) &\equiv 0 \pmod{48k + 22}, \\ \Rightarrow (j - i)(6k + 2) &\equiv 0 \pmod{24k + 11}, \not\equiv; \end{aligned} \quad (40)$$

$$(j - i)(12k + 4) \equiv 18k + 9 \pmod{48k + 22}, \not\equiv. \quad (41)$$

In addition, any two distinct differences x and y , produced by corresponding rows in V_2 and V_1 , satisfy $x + y \not\equiv 0 \pmod{48k + 22}$ as verified by Equation 42 for rows $2i$ and $2j$, Equation 43 for rows $2i + 1$ and $2j + 1$ and parity conditions in Equation 44 for rows $2i$ and $2j + 1$. In all such cases $x + y$ equals

$$\begin{aligned} (i + j + 1)(6k + 2) &\equiv 0 \pmod{24k + 11}, \\ \Rightarrow i + j + 1 &\equiv 0 \pmod{24k + 11}, \not\equiv; \end{aligned} \quad (42)$$

$$\begin{aligned} (i + j + 2)(6k + 2) &\equiv 0 \pmod{24k + 11}, \\ \Rightarrow i + j + 2 &\equiv 0 \pmod{24k + 11}, \not\equiv; \end{aligned} \quad (43)$$

$$(42k + 17) + (i + j)(12k + 4) \equiv 0 \pmod{48k + 22}, \not\equiv. \quad (44)$$

Respectively, the differences between entries in rows $2i$ and $2i + 1$ of V_3 and V_1 , are

$$\begin{aligned} 30k + 14 + i(12k + 7) - 2i &\equiv 30k + 14 + i(12k + 5) \pmod{48k + 22}, \\ 24k + 12 + i(12k + 7) - 2i - 1 &\equiv 24k + 11 + i(12k + 5) \pmod{48k + 22}. \end{aligned}$$

Equations 45 and 46 verify that these differences are all non-zero.

$$i \equiv (-30k - 14)(24k + 9) \equiv 12k + 6 \pmod{48k + 22}, \not\equiv; \quad (45)$$

$$i \equiv (-24k - 11)(24k + 9) \equiv 24k + 11 \pmod{48k + 22}, \not\equiv. \quad (46)$$

The differences are all distinct as verified by Equation 47 for rows $2i$ and $2j$, and rows $2i + 1$ and $2j + 1$, and using a parity argument in Equation 48 for rows $2i$ and $2j + 1$.

$$(j - i)(12k + 5) \equiv 0 \pmod{48k + 22}, \not\equiv; \quad (47)$$

$$(j - i)(12k + 5) \equiv 6k + 3 \pmod{48k + 22},$$

$$\Rightarrow j - i \equiv (6k + 3)(24k + 9) \equiv 12k + 5 \pmod{48k + 22}, \quad (48)$$

a contradiction, since $j = i + 12k + 5 > 12k + 4$.

In addition, any two distinct differences x and y , produced by corresponding rows in V_3 and V_1 , satisfy $x + y \not\equiv 0 \pmod{48k + 22}$ as verified by Equation 49 for rows $2i$ and $2j$, Equation 50 for rows $2i + 1$ and $2j + 1$ and Equation 51 for rows $2i$ and $2j + 1$. In all such cases $x + y$ equals

$$\begin{aligned} 12k + 6 + (i + j)(12k + 5) &\equiv 0 \pmod{48k + 22}, \\ \Rightarrow i + j \equiv (-12k - 6)(24k + 9) &\equiv 24k + 12 \pmod{48k + 22}, \not\equiv; \end{aligned} \quad (49)$$

$$(i + j)(12k + 5) \equiv 0 \pmod{48k + 22}, \not\equiv; \quad (50)$$

$$\begin{aligned} (6k + 3) + (i + j)(12k + 5) &\equiv 0 \pmod{48k + 22} \\ \Rightarrow i + j \equiv (-6k - 3)(24k + 9) &\equiv 36k + 17 \pmod{48k + 22}, \not\equiv. \end{aligned} \quad (51)$$

Respectively, the differences between entries are in rows $2i$ and $2i + 1$ of V_3 and V_2 , are

$$\begin{aligned} 30k + 14 - 30k - 13 + i(12k + 7 - 12k - 6) &\equiv i + 1 \pmod{48k + 22}, \\ 24k + 12 - 12k - 5 + i(12k + 7 - 12k - 6) &\equiv 12k + 7 + i \pmod{48k + 22}. \end{aligned}$$

These differences are all non-zero and distinct. In addition, any two distinct differences x and y satisfy $x + y \not\equiv 0 \pmod{48k + 22}$, By Lemmas 3.2, 3.3 and the above arguments the Latin squares \mathcal{L}_1 , \mathcal{L}_2 and \mathcal{L}_3 are cyclic MNOLSs. \square

4 Three cyclic MNOLSs of Order $48k + 38$, $k \geq 0$

In this section we construct two cyclic Latin squares \mathcal{L}_2 and \mathcal{L}_3 both of order $48k + 38$ and show that \mathcal{L}_1 (constructed by Example 1.3 with $n = 48k + 38$), \mathcal{L}_2 and \mathcal{L}_3 are cyclic MNOLSs. The following lemma is crucial in this section.

Lemma 4.1. *Let k be an integer. Working modulo $48k+38$, **1.** $\gcd(6k+5, 24k+19) = 1$; **2.** $\gcd(12k+11, 48k+38) = 1$; **3.** $\gcd(6k+4, 24k+19) = 1$; **4.** $\gcd(12k+9, 48k+38) = 1$.*

Proof. The following equations verify the statements given in the lemma: **1.** $4(6k+5) - (24k+19) = 1$; **2.** $(8k+7)(12k+11) - (2k+2)(48k+38) = 1$; **3.** $(8k+5)(6k+4) - (2k+1)(24k+19) = 1$; **4.** $(24k+17)(12k+9) - (6k+4)(48k+38) = 1$. \square

Working modulo $48k+38$ we define the $(24k+19) \times 1$ matrices (column vectors) $V_\alpha = [v_\alpha(i, 0)]$, $\alpha = 2, 3$, by

$$\begin{aligned} V_2 = & \{(2i, 0, 30k+23+i(12k+10)) \mid 0 \leq i \leq 12k+9\} \cup \\ & \{(2i+1, 0, 12k+9+i(12k+10)) \mid 0 \leq i \leq 12k+8\}, \end{aligned} \quad (52)$$

$$\begin{aligned} V_3 = & \{(2i, 0, 30k+24+i(12k+11)) \mid 0 \leq i \leq 12k+9\} \cup \\ & \{(2i+1, 0, 24k+20+i(12k+11)) \mid 0 \leq i \leq 12k+8\}. \end{aligned} \quad (53)$$

For $\alpha = 2, 3$, let $\mathcal{C}_\alpha = V_\alpha \cup \overline{V}_\alpha$, where

$$\overline{V}_\alpha = \{(48k+37-i, 0, 48k+37-v_\alpha(i, 0)) \mid 0 \leq i \leq 24k+18\}.$$

Note that \mathcal{C}_α has the reflection property. Now define $\mathcal{L}_\alpha = [l_\alpha(i, j)]$, where $l_\alpha(i, j) \equiv \mathcal{C}_\alpha(i, 0) + j \pmod{48k+38}$ for $0 \leq i, j \leq 48k+37$.

Lemma 4.2. *The array \mathcal{L}_2 is a Latin square of order $48k+38$, $k \geq 0$.*

Proof. The entries in V_2 are all distinct as verified by Equation 54 for the case rows $2i$ and $2j$, where $0 \leq i, j \leq 12k+9$, Equation 55 for the case rows $2i+1$ and $2j+1$, where $0 \leq i, j \leq 12k+8$, and Equation 56 for the case rows $2i$ and $2j+1$, where $0 \leq i \leq 12k+9$ and $0 \leq j \leq 12k+8$.

$$\begin{aligned} 30k+23+i(12k+10) & \equiv 30k+23+j(12k+10) \pmod{48k+38}. \\ \Rightarrow (j-i)(6k+5) & \equiv 0 \pmod{24k+19}, \not\equiv; \end{aligned} \quad (54)$$

$$\begin{aligned} 12k+9+i(12k+10) & \equiv 12k+9+j(12k+10) \pmod{48k+38}. \\ \Rightarrow (j-i)(6k+5) & \equiv 0 \pmod{24k+19}, \not\equiv; \end{aligned} \quad (55)$$

$$\begin{aligned} 30k+23+i(12k+10) & \equiv 12k+9+j(12k+10) \pmod{48k+38}, \\ \Rightarrow j-i & \equiv 4(9k+7) \pmod{24k+19}, \end{aligned} \quad (56)$$

implying $j = 12k+9+i > 12k+8$, which is a contradiction.

For any two rows containing entries x and y in V_2 , $x+y+1 \not\equiv 0 \pmod{48k+38}$, specifically Equation 57 for rows $2i$ and $2j$ of V_2 , Equation 58 for rows $2i+1$ and $2j+1$ and Equation 59 for rows $2i$ and $2j+1$.

$$\begin{aligned} 2(30k+23) + (i+j)(12k+10) + 1 & \equiv 0 \pmod{48k+38}, \\ \Rightarrow (i+j)(12k+10) & \equiv -12k-9 \pmod{48k+38}, \not\equiv; \end{aligned} \quad (57)$$

$$\begin{aligned} 2(12k+9) + (i+j)(12k+10) + 1 & \equiv 0 \pmod{48k+38}, \\ \Rightarrow (i+j)(12k+10) & \equiv -24k-19 \pmod{48k+38}, \not\equiv; \end{aligned} \quad (58)$$

$$\begin{aligned} 42k+32 + (i+j)(12k+10) + 1 & \equiv 0 \pmod{48k+38}, \\ \Rightarrow (i+j)(12k+10) & \equiv -42k-33 \pmod{48k+38}, \not\equiv. \end{aligned} \quad (59)$$

Thus the entries of \mathcal{C}_2 are all distinct and so \mathcal{L}_2 is a Latin square of order $48k + 38$. \square

Lemma 4.3. *The array \mathcal{L}_3 is a Latin square of order $48k + 38$, $k \geq 0$.*

Proof. The entries in V_3 are all distinct as verified by Equation 60 for the case rows $2i$ and $2j$, where $0 \leq i, j \leq 12k + 9$, Equation 61 for the case rows $2i + 1$ and $2j + 1$, where $0 \leq i, j \leq 12k + 8$, and Equation 62 for the case rows $2i$ and $2j + 1$, where $0 \leq i \leq 12k + 9$ and $0 \leq j \leq 12k + 8$.

$$\begin{aligned} 30k + 24 + i(12k + 11) &\equiv 30k + 24 + j(12k + 11) \pmod{48k + 38}, \\ \Rightarrow (j - i)(12k + 11) &\equiv 0 \pmod{48k + 38}, \not\equiv; \end{aligned} \quad (60)$$

$$\begin{aligned} 24k + 20 + i(12k + 11) &\equiv 24k + 20 + j(12k + 11) \pmod{48k + 38}, \\ \Rightarrow (j - i)(12k + 11) &\equiv 0 \pmod{48k + 38}, \not\equiv; \end{aligned} \quad (61)$$

$$\begin{aligned} 30k + 24 + i(12k + 11) &\equiv 24k + 20 + j(12k + 11) \pmod{48k + 38}, \\ \Rightarrow j - i &\equiv (8k + 7)(6k + 4) \equiv 36k + 28 \pmod{48k + 38}, \end{aligned} \quad (62)$$

implying $j = 36k + 28 + i > 12k + 8$ or $j = -12k - 10 + i$, which is a contradiction.

For any two rows containing entries x and y in V_3 , $x + y + 1 \not\equiv 0 \pmod{48k + 38}$, specifically Equation 63 for rows $2i$ and $2j$ of V_3 , Equation 64 for rows $2i + 1$ and $2j + 1$ and Equation 65 for rows $2i$ and $2j + 1$.

$$\begin{aligned} 2(30k + 24) + (i + j)(12k + 11) + 1 &\equiv 0 \pmod{48k + 38}, \\ \Rightarrow (i + j + 1)(12k + 11) &\equiv 0 \pmod{48k + 38}, \not\equiv \end{aligned} \quad (63)$$

$$\begin{aligned} 2(24k + 20) + (i + j)(12k + 11) + 1 &\equiv 0 \pmod{48k + 38}, \\ \Rightarrow i + j &\equiv (-3)(8k + 7) \equiv 24k + 17 \pmod{48k + 38}, \not\equiv; \end{aligned} \quad (64)$$

$$\begin{aligned} 54k + 44 + (i + j)(12k + 11) + 1 &\equiv 0 \pmod{48k + 38}, \\ \Rightarrow i + j &\equiv (-6k - 7)(8k + 7) \equiv 36k + 27 \pmod{48k + 38}, \not\equiv. \end{aligned} \quad (65)$$

Thus the entries of \mathcal{C}_3 are all distinct and so \mathcal{L}_3 is a Latin square of order $48k + 38$. \square

Theorem 4.4. *The Latin squares \mathcal{L}_1 , \mathcal{L}_2 and \mathcal{L}_3 are cyclic MNOLSs of order $48k + 38$, $k \geq 0$.*

Proof. Respectively, for rows $2i$ and $2i + 1$ the differences between entries of V_2 and V_1 , are

$$\begin{aligned} 30k + 23 + i(12k + 10) - 2i &\equiv 30k + 23 + i(12k + 8) \pmod{48k + 38}, \\ 12k + 9 + i(12k + 10) - 2i - 1 &\equiv 12k + 8 + i(12k + 8) \pmod{48k + 38}. \end{aligned}$$

These differences are all non-zero because in the first instance $30k + 23$ is odd but $12k + 8$ and $48k + 38$ are even and in the second if $(i + 1)(12k + 8) \equiv 0 \pmod{48k + 38}$, then by Lemma 4.1, $i + 1 \equiv 0 \pmod{24k + 19}$, which implies $i = 24k + 18$, a contradiction.

The differences are all distinct as verified by Equation 66 for rows $2i$ and $2j$ and for rows $2i + 1$ and $2j + 1$, and using parity conditions in Equation 67 for rows $2i$ and $2j + 1$.

$$(j - i)(6k + 4) \equiv 0 \pmod{24k + 19}, \not\equiv; \quad (66)$$

$$(j - i)(12k + 8) \equiv 18k + 15 \pmod{48k + 38}, \not\equiv. \quad (67)$$

In addition, any two distinct differences x and y , produced by corresponding rows in V_2 and V_1 , satisfy $x + y \not\equiv 0 \pmod{48k + 38}$ as verified by Equation 68 for rows $2i$ and $2j$, Equation 69 for $2i + 1$ and $2j + 1$ and using parity conditions together with Equation 70 for rows $2i$ and $2j + 1$. In all such cases $x + y$ is congruent to

$$(i + j + 1)(6k + 4) \equiv 0 \pmod{24k + 19}, \not\equiv; \quad (68)$$

$$(i + j + 2)(6k + 4) \equiv 0 \pmod{24k + 19}, \not\equiv; \quad (69)$$

$$(42k + 31) + (i + j)(12k + 8) \equiv 0 \pmod{24k + 19}, \not\equiv. \quad (70)$$

Respectively, for rows $2i$ and $2i + 1$ the differences between entries of V_3 and V_1 , are

$$\begin{aligned} 30k + 24 + i(12k + 11) - 2i &\equiv 30k + 24 + i(12k + 9) \pmod{48k + 38}, \\ 24k + 20 + i(12k + 11) - 2i - 1 &\equiv 24k + 19 + i(12k + 9) \pmod{48k + 38}. \end{aligned}$$

These differences are all non-zero because in the first instance if $30k + 24 + i(12k + 9) \equiv 0 \pmod{48k + 38}$, then by the proof of Lemma 4.1, $i \equiv (-30k - 24)(24k + 17) \equiv 12k + 10 \pmod{48k + 38}$, a contradiction and in the second instance if $24k + 19 + i(12k + 9) \equiv 0 \pmod{48k + 38}$, then by the proof of Lemma 4.1, $i \equiv (-24k - 19)(24k + 17) \equiv 24k + 19 \pmod{48k + 38}$, a contradiction.

The differences are all distinct as verified by Equation 71 for rows $2i$ and $2j$ and for rows $2i + 1$ and $2j + 1$, and Equation 72 for rows $2i$ and $2j + 1$.

$$(j - i)(12k + 9) \equiv 0 \pmod{48k + 38}, \not\equiv; \quad (71)$$

$$j - i \equiv (6k + 5)(24k + 17) \equiv 12k + 9 \pmod{48k + 38}, \quad (72)$$

implying $j = 12k + 9 + i > 12k + 8$, which is a contradiction.

Finally, for any two distinct differences x and y produced by corresponding rows in V_3 and V_1 , $x + y \not\equiv 0 \pmod{48k + 38}$ as verified by Equation 73 for rows $2i$ and $2j$, Equation 74 for $2i + 1$ and $2j + 1$ and parity conditions in Equation 75 for rows $2i$ and $2j + 1$. In all such cases $x + y$ is congruent to

$$\begin{aligned} 12k + 10 + (i + j)(12k + 9) &\equiv 0 \pmod{48k + 38}, \\ \Rightarrow i + j &\equiv (-12k - 10)(24k + 17) \equiv 24k + 20 \pmod{48k + 38}, \not\equiv; \end{aligned} \quad (73)$$

$$(i + j)(12k + 9) \equiv 0 \pmod{48k + 38}, \not\equiv; \quad (74)$$

$$\begin{aligned} 6k + 5 + (i + j)(12k + 9) &\equiv 0 \pmod{48k + 38}, \\ \Rightarrow i + j &\equiv (-6k - 5)(24k + 17) \equiv 36k + 29 \pmod{48k + 38}, \not\equiv. \end{aligned} \quad (75)$$

Respectively, the differences between entries are in rows $2i$ and $2i + 1$ of V_3 and V_2 , are

$$\begin{aligned} 30k + 24 - 30k - 23 + i(12k + 11 - 12k - 10) &\equiv i + 1 \pmod{48k + 38} \\ 24k + 20 - 12k - 9 + i(12k + 11 - 12k - 10) &\equiv 12k + 11 + i \pmod{48k + 38}. \end{aligned}$$

These differences are all non-zero and distinct. In addition, any two distinct differences x and y satisfy $x + y \not\equiv 0 \pmod{48k + 38}$.

By Lemmas 4.2, 4.3 and the above arguments, the Latin squares \mathcal{L}_1 , \mathcal{L}_2 and \mathcal{L}_3 are cyclic MNOLSs. \square

5 Three cyclic MNOLSs of Order $48k + 46$, $k \geq 0$

In this section we construct two cyclic Latin squares \mathcal{L}_2 and \mathcal{L}_3 both of order $48k + 46$ and show that \mathcal{L}_1 (constructed by Example 1.3 with $n = 48k + 46$), \mathcal{L}_2 and \mathcal{L}_3 are cyclic MNOLSs. The following lemma is crucial in this section.

Lemma 5.1. *Let k be an integer. Working modulo $48k + 46$, **1.** $\gcd(6k + 6, 24k + 23) = 1$; **2.** $\gcd(12k + 13, 48k + 46) = 1$; **3.** $\gcd(6k + 5, 24k + 23) = 1$; **4.** $\gcd(12k + 11, 48k + 46) = 1$.*

Proof. The following equations verify the statements given in the lemma: **1.** $4(6k + 6) - (24k + 23) = 1$; **2.** $(-8k - 7)(12k + 13) + (2k + 2)(48k + 46) = 1$; **3.** $(-8k - 9)(6k + 5) + (2k + 2)(24k + 23) = 1$; **4.** $(24k + 21)(12k + 11) + (6k + 6)(48k + 46) = 1$. \square

Working modulo $48k + 46$ we define the $(24k + 23) \times 1$ matrices (column vectors) $V_\alpha = [v_\alpha(i, 0)]$, $\alpha = 2, 3$, by

$$\begin{aligned} V_2 &= \{(2i, 0, 6k + 5 + i(12k + 12)) \mid 0 \leq i \leq 12k + 11\} \cup \\ &\quad \{(2i + 1, 0, 12k + 11 + i(12k + 12)) \mid 0 \leq i \leq 12k + 10\}, \end{aligned} \quad (76)$$

$$\begin{aligned} V_3 &= \{(2i, 0, 6k + 6 + i(12k + 13)) \mid 0 \leq i \leq 12k + 11\} \cup \\ &\quad \{(2i + 1, 0, 24k + 24 + i(12k + 13)) \mid 0 \leq i \leq 12k + 10\}. \end{aligned} \quad (77)$$

For $\alpha = 2, 3$, let $\mathcal{C}_\alpha = V_\alpha \cup \overline{V_\alpha}$, where

$$\overline{V_\alpha} = \{(48k + 45 - i, 0, 48k + 45 - c_\alpha(i, 0)) \mid 0 \leq i \leq 24k + 22\}.$$

Note that \mathcal{C}_α has the reflection property. Now define $\mathcal{L}_\alpha = [l_\alpha(i, j)]$, where $l_\alpha(i, j) \equiv \mathcal{C}_\alpha(i, 0) + j \pmod{48k + 46}$ for $0 \leq i, j \leq 48k + 45$.

Lemma 5.2. *The array \mathcal{L}_2 is a Latin square of order $48k + 46$, $k \geq 0$.*

Proof. The entries in V_2 are all distinct as verified by Equation 78 for the case rows $2i$ and $2j$, where $0 \leq i, j \leq 12k + 11$, Equation 79 for rows $2i + 1$ and $2j + 1$,

where $0 \leq i, j \leq 12k + 10$, and Equation 80 for the case rows $2i$ and $2j + 1$, where $0 \leq i \leq 12k + 11$ and $0 \leq j \leq 12k + 10$.

$$\begin{aligned} 6k + 5 + i(12k + 12) &\equiv 6k + 5 + j(12k + 12) \pmod{48k + 46}, \\ \Rightarrow (j - i)(6k + 6) &\equiv 0 \pmod{24k + 23}, \not\equiv; \end{aligned} \quad (78)$$

$$\begin{aligned} 12k + 11 + i(12k + 12) &\equiv 12k + 11 + j(12k + 12) \pmod{48k + 46}, \\ \Rightarrow (j - i)(6k + 6) &\equiv 0 \pmod{24k + 23}, \not\equiv; \end{aligned} \quad (79)$$

$$\begin{aligned} 6k + 5 + i(12k + 12) &\equiv 12k + 11 + j(12k + 12) \pmod{48k + 46}, \\ \Rightarrow j - i &\equiv 4(-3k - 3) \pmod{24k + 23}, \not\equiv. \end{aligned} \quad (80)$$

implying $j = 12k + 11 + i > 12k + 10$, which is a contradiction.

For any two rows containing entries x and y in V_2 , parity conditions and the following equations can be used to verify $x + y + 1 \not\equiv 0 \pmod{48k + 46}$, specifically Equation 81 for rows $2i$ and $2j$, Equation 82 for rows $2i + 1$ and $2j + 1$ and Equation 83 for rows $2i$ and $2j + 1$.

$$\begin{aligned} 2(6k + 5) + (i + j)(12k + 12) + 1 &\equiv 0 \pmod{48k + 46}, \\ \Rightarrow (i + j)(12k + 12) &\equiv -12k - 11 \pmod{48k + 46}, \not\equiv; \end{aligned} \quad (81)$$

$$\begin{aligned} 2(12k + 11) + (i + j)(12k + 12) + 1 &\equiv 0 \pmod{48k + 46}, \\ \Rightarrow (i + j)(12k + 12) &\equiv -24k - 23 \pmod{48k + 46}, \not\equiv; \end{aligned} \quad (82)$$

$$\begin{aligned} 18k + 16 + (i + j)(12k + 12) + 1 &\equiv 0 \pmod{48k + 46}, \\ \Rightarrow (i + j)(12k + 12) &\equiv -18k - 17 \pmod{48k + 46}, \not\equiv. \end{aligned} \quad (83)$$

Thus the entries of \mathcal{C}_2 are all distinct and so \mathcal{L}_2 is a Latin square of order $48k + 46$. \square

Lemma 5.3. *The array \mathcal{L}_3 is a Latin square of order $48k + 46$, $k \geq 0$.*

Proof. The entries in V_3 are all distinct as verified by Equation 84 for the case rows $2i$ and $2j$, where $0 \leq i, j \leq 12k + 11$, Equation 85 for the case rows $2i + 1$ and $2j + 1$, where $0 \leq i, j \leq 12k + 10$, and Equation 86 for the case rows $2i$ and $2j + 1$, where $0 \leq i \leq 12k + 11$ and $0 \leq j \leq 12k + 10$.

$$\begin{aligned} 6k + 6 + i(12k + 13) &\equiv 6k + 6 + j(12k + 13) \pmod{48k + 46}, \\ \Rightarrow (j - i)(12k + 13) &\equiv 0 \pmod{48k + 46}, \not\equiv; \end{aligned} \quad (84)$$

$$\begin{aligned} 24k + 24 + i(12k + 13) &\equiv 24k + 24 + j(12k + 13) \pmod{48k + 46}, \\ \Rightarrow (j - i)(12k + 13) &\equiv 0 \pmod{48k + 46}, \not\equiv; \end{aligned} \quad (85)$$

$$\begin{aligned} 6k + 6 + i(12k + 13) &\equiv 24k + 24 + j(12k + 13) \pmod{48k + 46}, \\ \Rightarrow j - i &\equiv (-8k - 7)(-18k - 18) \equiv 36k + 34 \pmod{48k + 46}, \end{aligned} \quad (86)$$

implying $j = 36k + 34 + i > 12k + 10$ or $j = -12k - 12 + i < 0$, which is a contradiction.

For any two rows containing entries x and y in V_3 , $x + y + 1 \not\equiv 0 \pmod{48k + 46}$ as verified by Equation 87 for rows $2i$ and $2j$, Equation 88 for rows $2i + 1$ and $2j + 1$

and Equation 89 for rows $2i$ and $2j + 1$.

$$\begin{aligned} 2(6k + 6) + (i + j)(12k + 13) + 1 &\equiv 0 \pmod{48k + 46}, \\ \Rightarrow (i + j + 1)(12k + 13) &\equiv 0 \pmod{48k + 46}, \not\equiv; \end{aligned} \quad (87)$$

$$\begin{aligned} 2(24k + 24) + (i + j)(12k + 13) + 1 &\equiv 0 \pmod{48k + 46}, \\ \Rightarrow i + j \equiv (-3)(-8k - 7) &\equiv 24k + 21 \pmod{48k + 46}, \not\equiv; \end{aligned} \quad (88)$$

$$\begin{aligned} 30k + 30 + (i + j)(12k + 13) + 1 &\equiv 0 \pmod{48k + 46}, \\ \Rightarrow i + j \equiv (-30k - 31)(-8k - 7) &\equiv 36k + 33 \pmod{48k + 46}, \not\equiv. \end{aligned} \quad (89)$$

Thus the entries of \mathcal{C}_3 are all distinct and so \mathcal{L}_3 is a Latin square of order $48k + 46$. \square

Theorem 5.4. *The Latin squares \mathcal{L}_1 , \mathcal{L}_2 and \mathcal{L}_3 are cyclic MNOLSs of order $48k + 46$, $k \geq 0$.*

Proof. Respectively, the differences between entries in rows $2i$ and $2i + 1$ of V_2 and V_1 , are

$$\begin{aligned} 6k + 5 + i(12k + 12) - 2i &\equiv 6k + 5 + i(12k + 10) \pmod{48k + 46}, \\ 12k + 11 + i(12k + 12) - 2i - 1 &\equiv 12k + 10 + i(12k + 10) \pmod{48k + 46}. \end{aligned}$$

These differences are all non-zero since in the first instance 2 divides $12k + 10$ and $48k + 46$ but does not divide $6k + 5$ and in the second if $(i + 1)(12k + 10) \equiv 0 \pmod{48k + 46}$, then by Lemma 5.1, $i + 1 \equiv 0 \pmod{24k + 23}$, which implies $i = 24k + 12$, a contradiction.

The differences are all distinct as verified by Equation 90 for rows $2i$ and $2j$ and for rows $2i + 1$ and $2j + 1$, and parity conditions in Equation 91 for rows $2i$ and $2j + 1$.

$$\begin{aligned} (j - i)(12k + 10) &\equiv 0 \pmod{48k + 46}, \\ \Rightarrow (j - i)(6k + 5) &\equiv 0 \pmod{24k + 23}, \not\equiv; \end{aligned} \quad (90)$$

$$(j - i)(12k + 10) \equiv -6k - 5 \pmod{48k + 46}, \not\equiv. \quad (91)$$

In addition, any two different rows the two differences x and y , produced by corresponding rows of V_2 and V_1 , satisfy $x + y \not\equiv 0 \pmod{48k + 46}$, as verified by Equation 92 for rows $2i$ and $2j$, Equation 93 for rows $2i + 1$ and $2j + 1$ and parity conditions in Equation 94 for rows $2i$ and $2j + 1$. In all such cases $x + y$ is congruent to

$$(i + j + 1)(6k + 5) \equiv 0 \pmod{24k + 23}, \not\equiv; \quad (92)$$

$$(i + j + 2)(6k + 5) \equiv 0 \pmod{24k + 23}, \not\equiv; \quad (93)$$

$$(18k + 15) + (i + j)(12k + 10) \equiv 0 \pmod{48k + 46}, \not\equiv. \quad (94)$$

Respectively, the differences between entries in rows $2i$ and $2i + 1$ of V_3 and V_1 , are

$$\begin{aligned} 6k + 6 + i(12k + 13) - 2i &\equiv 6k + 6 + i(12k + 11) \pmod{48k + 46}, \\ 24k + 24 + i(12k + 13) - 2i - 1 &\equiv 24k + 23 + i(12k + 11) \pmod{48k + 46}. \end{aligned}$$

Equations 95 and 96 verify that these differences are all non-zero.

$$i \equiv (-6k - 6)(24k + 21) \equiv 12k + 12 \pmod{48k + 46}, \not\equiv; \quad (95)$$

$$i \equiv (-24k - 23)(24k + 21) \equiv 24k + 23 \pmod{48k + 46}, \not\equiv. \quad (96)$$

If two differences produced by rows $2i$ and $2j$ or by rows $2i + 1$ and $2j + 1$ are equal, then $(j - i)(12k + 11) \equiv 0 \pmod{48k + 46}$. Now by Lemma 5.1, $i = j$. Equation 97 verifies that two differences produced by rows $2i$ and $2j + 1$ are never equal.

$$\begin{aligned} (j - i)(12k + 11) &\equiv -18k - 17 \pmod{48k + 46}, \\ \Rightarrow j - i &\equiv (-18k - 17)(24k + 21) \equiv 12k + 11 \pmod{48k + 46}, \end{aligned} \quad (97)$$

implying $j = 12k + 11 + i > 12k + 10$, which is a contradiction.

In addition, any two different rows the two differences x and y , produced by corresponding rows of V_3 and V_1 , satisfy $x + y \not\equiv 0 \pmod{48k + 46}$, as verified by Equation 98 for rows $2i$ and $2j$, Equation 99 for rows $2i + 1$ and $2j + 1$ and parity arguments together with Equation 100 for rows $2i$ and $2j + 1$. In all such cases $x + y$ is congruent to

$$\begin{aligned} 12k + 12 + (i + j)(12k + 11) &\equiv 0 \pmod{48k + 46}, \\ \Rightarrow i + j &\equiv (-12k - 12)(24k + 21) \equiv 24k + 24 \pmod{48k + 46}, \not\equiv; \end{aligned} \quad (98)$$

$$(i + j)(12k + 11) \equiv 0 \pmod{48k + 46}, \not\equiv; \quad (99)$$

$$\begin{aligned} 30k + 29 + (i + j)(12k + 11) &\equiv 0 \pmod{48k + 46}, \\ \Rightarrow i + j &\equiv (-30k - 29)(24k + 21) \equiv 36k + 35 \pmod{48k + 46}, \not\equiv. \end{aligned} \quad (100)$$

Respectively, the differences between entries are in rows $2i$ and $2i + 1$ of V_3 and V_2 , are

$$\begin{aligned} 6k + 6 - 6k - 5 + i(12k + 13 - 12k - 12) &\equiv i + 1 \pmod{48k + 46}, \\ 24k + 24 - 12k - 11 + i(12k + 13 - 12k - 12) &\equiv 12k + 13 + i \pmod{48k + 46}. \end{aligned}$$

These are all non-zero and distinct. In addition, any two distinct differences x and y satisfy $x + y \not\equiv 0 \pmod{48k + 46}$.

By Lemmas 5.2, 5.3 and the above arguments, the Latin squares \mathcal{L}_1 , \mathcal{L}_2 and \mathcal{L}_3 are cyclic MNOLSs. \square

References

- [1] C.J. Colbourn and J.H. Dinitz, (Eds.), Handbook of combinatorial designs. CRC press, 2010.
- [2] A.B. Evans, Orthomorphism graphs of groups, Lecture Notes in Mathematics, volume 1535, Springer-Verlag, 1992.
- [3] P.C. Li and G.H.J. van Rees, *Nearly Orthogonal Latin Squares* Journal of Combinatorial Mathematics and Combinatorial Computing **62** (2007), 13–24.

- [4] D. Raghavarao, S.S. Shrikhande and M.S. Shirkhande, *Incidence Matrices and Inequalities for Combinatorial Designs* Journal of Combinatorial Design **10** (2002), 17–26.
- [5] D.T. Todorov, *Four Mutually Orthogonal Latin Squares of Order 14* Journal of Combinatorial Design **20** (2012), 363-367.
- [6] I.M. Wanless, *Atomic Latin squares based on cyclotomic orthomorphisms*, Electronic Journal of Combinatorics **12** (2005), R22.